Investigating the Chip-Firing Problem and the Sandpile Group

McNair Research

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Abstract

It is known that each combinatorial graph determines an abelian group called its sandpile group. The primary objective of this research is to examine how the graph determines the properties of the corresponding sandpile group.

This investigation first focused on a graph that is a cycle. Secondly, it studied a variation of the cycle, delving into the mathematical properties and dynamics of sandpile groups within these structures. Through analysis, consistent generators and identities were found and proven for these sandpile groups.

Introduction

The chip-firing problem and the sandpile group are recently developed areas of study in mathematics, offering insights into real-world phenomena such as the formation of sand dunes. These problems are characterized by their ability to generate complex behavior from simple local rules, making them captivating subjects for investigation.

In 1987, the chip-firing problem was introduced as an illustration of self-organized criticality, a concept that characterizes natural processes gravitating toward a delicate equilibrium that can easily become unstable [5].

The problem uses a graph to represent a configuration of sandpile locations, with each pile assigned an integer value corresponding to the number of grains of sand it holds. The dynamics of the system are driven by the heights of the sandpiles; a sandpile that is too high is unstable and can topple, thereby distributing some of the grains of sand at its location to neighboring locations. When a pile topples, it will distribute a minimal amount of sand to its neighbors according to its out-degree, the number of adjacent piles in the examples we will study.

In this research, we focus on examining the characteristics of the sandpile group for a given arrangement of sandpiles.

By delving into the properties and relationships within these sandpile groups, we hope to shed light on the fundamental characteristics of the sandpile group of a graph.

Figure 1: An example of a graph representation of a chip-firing system with four piles in a line.

A pile is *unstable* if its value exceeds the number of its neighboring piles. When a pile is unstable, it *fires*, distributing one of its values to each of its neighbors. This redistribution of value may lead to the firing of neighboring piles, the process continues repeatedly. Either this process continues forever or it reaches a *stable* distribution where no more firing can occur. One way to guarantee that we reach a stable distribution is to assume that the underlying structure has a *sink* and is connected, i.e., one can get from any pile to any other by moving between a sequence of adjacent piles. The sink serves as a recipient for chips from neighboring piles but does not contribute any value itself. In other words, when a pile fires, any chips destined for the sink will "fall to the floor" and not be tracked further.

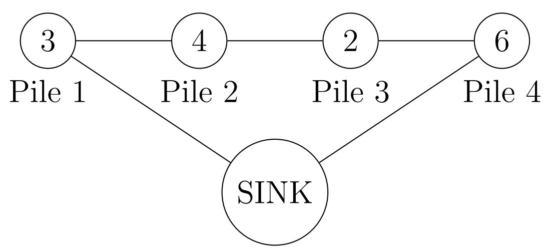

Figure 2: The chip-firing system from Figure 1 with added values to each pile a sink

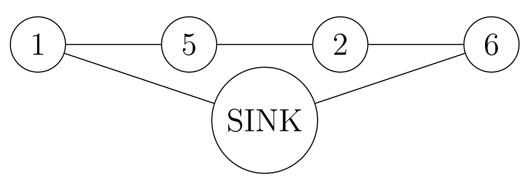

Figure 3: The chip-firing system from Figure 2 after Pile 1 fires.

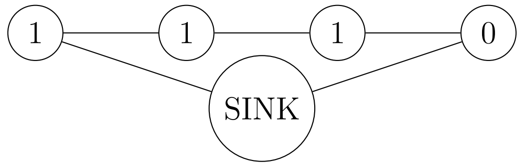

Figure 4: The system from Figure 2 after firing until a stable configuration was reached.

In our research, we will represent the configuration in Figure 4 as (1110) or $(1,1,1,0)$. We may also refer to this as the *sandpile circular configuration of size n.* The set of all stable and *reachable* configurations of piles is known as *recurrent*, these recurrent configurations form a set. For the previous example system, the recurrent elements are known to be $(1,1,1,1),(0,1,1,1),(1,0,1,1),(1,1,0,1)$, and $(1,1,1,0)$.

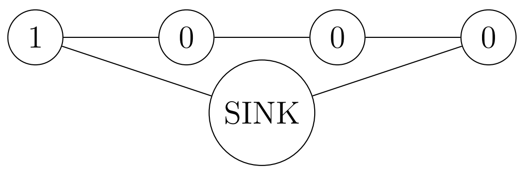

Figure 5: This is an example of a stable yet unreachable configuration as it cannot be reached from any other configuration.

This set possesses an *identity* element. When the identity element is added to any other element, and the firing process continues until a stable configuration is reached, the resulting element will be that other element. For instance, for the graph in *Figure* 2, the identity element is $(1,1,1,1)$. Therefore, if we take the result from *Figure* 4 and add it to the identity element, $(1,1,1,0)+(1,1,1,1) = (2,2,2,1)$, which then fires down to $(1,1,1,0)$. In fact, any two recurrent sandpile configurations added together and fired until stable will result in one of the recurrent configurations.

Because addition is associative, so is the addition of recurrent elements. For example, $(1,1,1,0) + (1,1,1,1)) + (1,1,0,1) = (2,2,2,1) + (1,1,0,1) = (3,3,2,2) = (1,0,1,1)$ and $(1,1,1,0)+((1,1,1,1)+(1,1,0,1)) = (1,1,1,0)+(2,2,1,2) = (3,3,2,2) = (1,0,1,1)$, these are equivalent.

Another property is that each element s of the set has an inverse s^{-1}, that is $s+s^{-1} = I$ where I is the identity of the set. For example, with $s = (1,1,0,1)$, $s^{-1} = (1,0,1,1)$ because

$(1,1,0,1)+(1,0,1,1)=(2,1,1,2)=(1,1,1,1)$ which is the identity.

Because of these characteristics, the set forms a mathematical object known as a *group*. We call the set of recurrent configurations with the above operation the *sandpile group* of the graph. Because the addition of any elements in this set is commutative ($a+b=b+a$ for any set elements a and b) the sandpile group is a special kind of group called an *abelian group*.

Methods

The research methods employed in this study involved the analysis of sandpile groups using the SageMath computer system [10] and the application of various proof techniques, mainly direct proofs, and proofs by induction.

SageMath software was utilized for general computations and computations related to the sandpile group, as well as other aspects of group theory and linear algebra [10]. This system provides packages specifically designed for studying sandpile groups, making it a suitable tool for our research. Through this tool, we were able to study certain configurations of hypothetical patterns and ultimately prove our hypotheses.

To investigate the characteristics of sandpile groups and their relationship to graph properties, we conducted analyses using sandpile configurations of different sizes. Specifically, we focused on graphs with a sink that acts as a pile in the system but does not actively participate in the value distribution.

Throughout the research, a combination of computational analysis and rigorous mathematical proofs was employed to investigate the sandpile group and its relation to graph properties. These methods allowed us to analyze different sandpile configurations, explore their properties, and establish theorems and conjectures.

Theorem 1. *The following is true for the sandpile group of $n+1$ vertices in a circle, with one vertex as the sink.*

(a) *The sandpile group is cyclic*

(b) *If* $n = 2k$ *is even, than one generator* a *is the vector of ones except zero in position* k*; and another generator* b *is given by the vector of ones except zero in position* $(k+1)$.

(c) *If* $n = 2k+1$ *is odd, then one generator* a *is a vector of ones except a zero in position* k*, another generator* b *is given by the vector of ones except a zero in position* $(k+2)$.

Proof. Let a denote the sandpile group element of ones with a zero in position k. The one's vector, $(1, 1, \cdots, 1)$ is the maximal stable configuration. We will denote this I. Now, consider the operation $a + I = (1, \cdots, 1, 0, 1, \cdots, 1) + (1, \cdots, 1) = (2, \cdots, 2, 1, 2, \cdots, 2)$. After firing down, we get the sequence $(2, \cdots, 2, 1, 2, \cdots, 2) \to (1, 3, 2, \cdots, 2, 2, 0, 3, 2, \cdots, 2) \to (2, 1, 3, \cdots, 2, 2, 1, 1, 3, \cdots, 2) \to (2, 1, 3, \cdots, 2, 1, 2, \cdots, 2)$.

Continuing this process, it eventually fires down to $(1, \cdots, 1, 0, 1, \cdots, 1) = a$. Thus, a is recurrent [6].

The Laplacian matrix L with the sink row and column removed, is denoted as $\hat{L}$ for the chip-firing configuration.

By Kirchhoff's Theorem, if L denotes the Laplacian matrix of a graph G then the number of spanning trees in G is equal to $(-1)^{i+j} \det L(i,j)$ where the i^{th} row and j^{th} column are removed from L to obtain $L(i,j)$ [4]. In our case $i = j$ so the number of spanning trees in G is equal to $\det \hat{L}$. Because the sink is removed from our graph, the remainder forms a spanning tree and thus the determinant is nonzero. Hence $\hat{L}$ is invertible.

Now, consider the inverse of $\hat{L}$. The first column of $\hat{L}^{-1}$ is the solution to the equation:

$$\hat{L} \cdot x = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = e_1.$$

Let $\hat{L}_{i \leftarrow e_i}$ denote the matrix $\hat{L}$ with the i-th column replaced by the i-th column of the identity matrix. By Cramer's rule, we find that the ith entry of x is given by:

$$x_i = \frac{\det(\hat{L}_{i \leftarrow e_i})}{\det(\hat{L})} = \frac{n-i+1}{n+1}.$$

Hence, the first row of $\hat{L}^{-1}$ can be expressed as $\frac{1}{n+1} \cdot [n \ \cdots \ 2\,1]$. Consequently, the result of multiplying the first row of $\hat{L}^{-1}$ by the vector a yields:

$$\frac{1}{n+1} \cdot [n \ \cdots \ 2\,1] \cdot a = \frac{(n+1)(k-1)+k+1}{n+1} = (k-1) + \frac{k+1}{2k+1}.$$

Note that $\frac{k+1}{2k+1}$ is a reduced fraction. d. To demonstrate this, note that $2k+1 = 2(k+1)-1$ so $2(k+1)-1(2k+1) = 1$.By Bezout's Identity, we observe that the greatest common divisor (gcd) of $2k+1$ and $k+1$ is 1. This implies that they are coprime, as the only common divisor is 1.

Thus the order of a is the least common multiple of the denominators of the elements of x, and the order of x must be less than or equal to $n+1$ we conclude that the order of v_a is $2k+1 = n+1$. Thus, a is a generator. A similar proof can be used to show that if n is odd, i.e., $n = 2k+1$ for some integer k, a defined as the vector of ones with a zero in the k^{th} position is also a generator.

Similarly, we can show that the vector of ones with a zero in position $(k+1)$ also has an order of $n+1$ and is therefore also a generator for both the even case. The vector of ones with a zero in position $(k+2)$ will be a generator of the sandpile group in the odd n case. □

Theorem 2. *Let S be the configuration consisting of a path of n piles and an additional pile that is a sink and is connected to each vertex in the path. Then*

(a) *S is cyclic*

(b) *One generator, a, is the vector of ones except a zero in the first position. A second generator, b, is the vector of ones except a zero in its n^{th} position.*

Proof. Consider the vector a defined as a vector of ones with a zero in the first position.

We know that the identity I of S is the all-ones vector. Now consider $a + I$. This will look like $(0, 1, 1, \dots, 1) + (0, 1, 1, \dots, 1) = (1, 2, 2, \dots, 2)$. In this configuration, only the 2 in the nth position can topple, so $(1, 2, 2, \dots, 2) \to (1, 2, 2, \dots, 3, 0)$. Now only the 3 in the $(k-1)$th position can topple, so $(1, 2, 2, \dots, 3, 0) \to (1, 2, 2, \dots, 3, 0, 1)$. This pattern will continue: $(1, 2, 2, \dots, 3, 0, 1) \to (1, 2, 2, \dots, 3, 0, 1, 1) \to \cdots \to (0, 1, 1, \dots, 1) = a$. Thus, a is recurrent.

Let $\hat{L}$ represent the Laplacian matrix obtained by removing the sink row and column from the configuration's Laplacian matrix L.

According to Krichoff's Theorem, if L denotes the Laplacian matrix of a graph G, the number of spanning trees in G can be determined by $(-1)^{i+j} \det \hat{L}$, where $\hat{L}$ is obtained by removing the ith row and jth column from L [4]. In our case, since $i = j$, the number of spanning trees in G is simply $\det \hat{L}$. As the sink is removed from our graph, the remaining structure forms a spanning tree, ensuring that the determinant is non-zero. Therefore $\hat{L}$ is invertible.

$$\hat{L} = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 3 & -1 & \dots & 0 & 0 \\ 0 & -1 & 3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 3 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}$$

Note that $\hat{L}$ is an $(n \times n)$ matrix, where $n \geq 2$. For $n = 1$, L is a 1×1 matrix with determinant 1. For $n = 2$, $\hat{L}$ is a 2×2 matrix with determinant 3.

Using cofactor expansion on the first column to find $det\hat{L}$, we get $det\hat{L} = \hat{L}(1|1) + \hat{L}(2|1) + \cdots = 2 * \det A_{n-1} - \det A_{n-2}$ where A_n is an nxn matrix that takes the form

$$A_n = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 3 & -1 & \dots & 0 & 0 \\ 0 & -1 & 3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 3 & -1 \\ 0 & 0 & 0 & \dots & -1 & 3 \end{bmatrix}$$

Because this is a tri-diagonal matrix, the n-th continuant K_n is defined recursively as $K_n =$

$3 * K_{n-1} - K_{n-2}$ with the initial conditions $K_1 = 1$ and $K_2 = 3$. It's important to note that using this relation K_n is equivalent to the $2n - 1^{st}$ Fibonacci number. Now we have $\det A = \text{Fib}_{(2*n-1)}$ So $\det \hat{L} = 2 * \text{Fib}_{(2*n-1)-1} - \text{Fib}_{(2*n-1)-1} = 2 * \text{Fib}_{2*n-3} - \text{Fib}_{2*n-5} = \text{Fib}_{2*(n-1)}$

This is equivalent to a bisection of the Fibonacci numbers, $d_(n) = \text{Fib}_{2(n-1)}$. It is known that the Fibonacci numbers are a strongly divisible sequence, meaning that $gcd(\text{Fib}_n, \text{Fib}_m) = \text{Fib}_{gcd(n,m)}$. Now consider $d(n)$ and $d(n+1)$. We know that $d_n = \text{Fib}_{2(n-1)}$ and $d_{n+1} = \text{Fib}_{2((n+1)-1)} = \text{Fib}_{2n}$. Since n and $n+1$ are coprime, we have $gcd(2n, 2(n+1)) = 2$. Therefore, $\gcd(\text{Fib}_{2n}, \text{Fib}_{2(n+1)}) = \text{Fib}_{\gcd(2(n-1),2n} = \text{Fib}_2 = 1$. Thus, Fib_{2n} and $\text{Fib}_{2(n-1)}$, and consequently $d(n)$ and $d(n+1)$, are coprime.

Consider the solution to the equation $\hat{L} \cdot x = a$. By applying Cramer's rule, the i-th entry of x is given by

$$x_i = \frac{\det(\hat{L}_{i \to a})}{\det(\hat{L})} = \frac{\text{Fib}_{2(n-i-1)+1}}{\text{Fib}_{2(n-1)}},$$

Since $\gcd(2(n-i-1)+1, 2(n-1)) = 1$, according to the strong divisibility principle of the Fibonacci numbers, we have $\gcd(\text{Fib}_{2(n-i-1)+1}, \text{Fib}_{2(n-1)}) = \text{Fib}_1 = 1$. Thus, the numerator and denominator of any x_i are coprime. Therefore, each x_i is reduced.

Consequently, because the order of a is the least common multiple of the denominators of the elements of x, and the order of x must be less than or equal to Fib_{2n}, we conclude that the order of a is $\text{Fib}_{2(n-1)}$, implying that a is a generator. A similar proof can be used to show that a vector of ones with a zero in the last position is also a generator. □

Conclusion

This research delved into the intriguing world of the Chip-Firing Problem and its associated mathematical object, the Sandpile Group. We explored multiple sandpile configuration families, analyzing their dynamics and uncovering enthralling properties.

Through our investigation, we made discoveries regarding the structure of the sandpile group for specific pile arrangements. Notably, for the families we studied, we ascertained that the sandpile groups are cyclic and have two generators, represented by vectors with zeros in specific positions.

The Chip-Firing Problem and Sandpile Group continue to be alluring areas for future research.

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